Feature selection and volatility modeling of European options

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European option

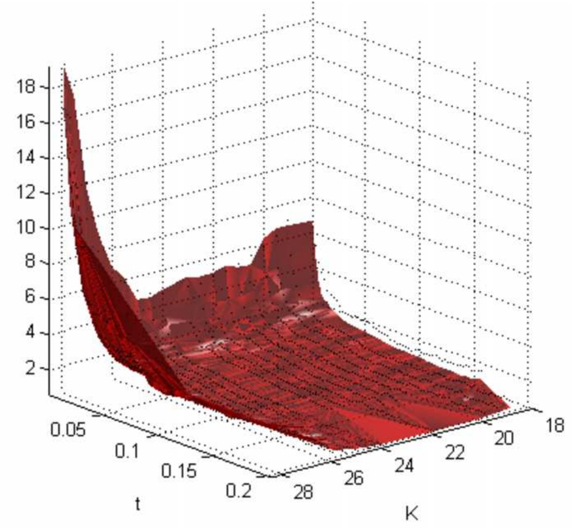
- The option is an instrument that conveys the right, but not the obligation, to engage in a future transaction on some underlying security.
- European option is an option that may only be exercised on expiration

Implied volatility:

$$\sigma = arg \min_{\sigma \in [0,1.5]} \left(C_{K,t} - C(\sigma, P_t, B, K, t) \right).$$

- **K** is strike price,
- t is the set of time ticks,
- B is risk-free rate,
- $C_{K,t}$ is the historical price,
- P_t is the historical security price.

Volatility smile model



Background

- The set of strike prices, time ticks and volatility values is given.
- Feature generation is necessary.
- The number of the features exceeds the number of the objects.
- Feature selection is a must.

Sample set $\xrightarrow{\text{Feature generation}}$ **Expanded set of features** $\xrightarrow{\text{Feature selection}}$ **Resulting Model**

Feature generation

Let

- $\Xi = \{\xi^u\}_{u=1}^U$ be the set of free variables;
- $G = \cup\{g_v\}_{v=2}^V$ be the finite set of primitives given by experts.
  For example $G = \left\{\frac{1}{x}, \sqrt{x}, \ln(x), \tanh(x)\right\}$.

Put $a_\iota = g_v(\xi^u)$ determined by $\iota = (v-1)U + u$.
Put $x_j = \prod a_{i_1} a_{i_2} \dots a_{i_s}$,

$i_1, i_2, \dots, i_s \in \{1, 2, \dots, UV\}$ and $s = 1, 2, \dots, R$.

$$\xi_u \xrightarrow{g_v} g_v(\xi_u) \stackrel{\text{def}}{=} a_\iota \xrightarrow{\prod} x_j$$

Problem statement

The sample:

$$\{(\mathbf{x}^i, y^i) | i = 1, \dots, m\}, \quad \mathbf{x}^i \in \mathbb{R}^n, y^i \in \mathbb{R}^1, \; n = |N|, \; N \subset \mathbb{N}.$$

The design matrix: $X = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$.

The set of indices $\mathcal{A} \subset \mathcal{Z} = \{1, 2, \dots, n\}$ corresponding the optimal model is required.
The optimal model:

$$y^i = \sum_{j \in \mathcal{A}} x_j^i w_j,$$

$$S = \sum_{i=1}^{m} \left(\sum_{j \in \mathcal{A}} x_j^i w_j - y^i \right)^2 \to \min .$$

Bayesian model selection

1. Estimation of model parameters.
2. Model evidence is $p(D|f_i)$.
   The likelihood function $p(f_i|D)$ is given by Bayes' formula:

$$p(f_i|D) = \frac{\mathbf{p(D|f_i)}p(f_i)}{p(D)}.$$

   For all i, j, $p(f_i) = p(f_j)$.
   Comparison of models:

$$\frac{p(f_i|D)}{p(f_j|D)} = \frac{p(D|f_i)}{p(D|f_i)}.$$

Evidence calculation

The likelihood function

$$p(D|\mathbf{w}, f, \beta) = \prod_{i=1}^{m} \mathcal{N}(y^i | f(\vec{x}^i, \mathbf{w}), \beta^{-1}).$$

Let $\mathbf{w} \sim \mathcal{N}(0, \alpha I)$.
The evidence is given by:

$$p(D|f, w, \alpha, \beta) = \int p(D|\mathbf{w}, f, \beta) p(\mathbf{w}|f, \alpha) d\mathbf{w}.$$

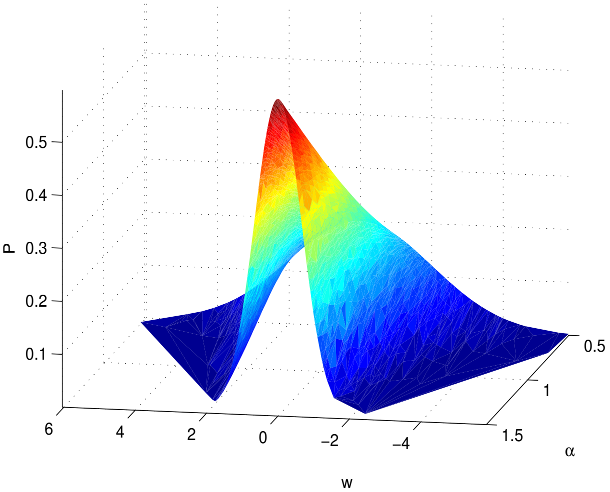
Calculation:

$$\ln p(D|f, w, \alpha, \beta) = -\frac{w}{2}\|\mathbf{y} - X\mathbf{w}\|^2 - \frac{\alpha}{2}\mathbf{w}^T\mathbf{w} - \frac{1}{2}\ln H +$$

$$+\frac{m}{2}\ln\beta + \frac{l}{2}\ln\alpha - \frac{m}{2}\ln 2\pi,$$

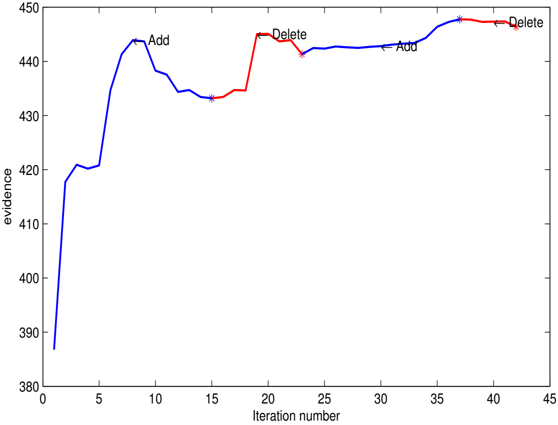
l is the number of parameters in the model.

An illustration of parameter distribution depending on α



Selection of the most evident model

The evidence from Coherent Bayesian Inference is maximizing:
1. **Add** 2. **Delete** features, while the evidence value is increasing and some steps while decreasing.



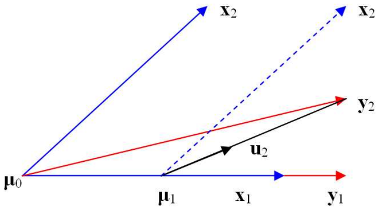
Algorithm of evidence maximization

Set of all features indexes $\mathcal{Z} = \{1, 2, \dots, n\}$.
The current set $\mathcal{A}_k$. $\mathcal{A}_0 = \emptyset$.
Consider k-th algorithm step.

1. Adding: $\mathcal{A}_k = \mathcal{A}_{k-1}$; features are added from $\mathcal{Z} \setminus \mathcal{A}_{k-1}$ into $\mathcal{A}_k$ by criterion $\mathcal{C}_L$, while criterion $\mathcal{C}_\mathcal{E}$ is completed.
2. Deleting: features are deleted by criterion $\mathcal{C}_D$, while criterion $\mathcal{C}_\mathcal{E}$ is completed.

- $\mathcal{C}_L$: LARS step.
- $\mathcal{C}_D$: Belsley method.
- $\mathcal{C}_\mathcal{E}$: evidence is not less then $\mathcal{E}_{min}$.

1. Add a feature: Least angle regression (LARS)



Put $\boldsymbol{\mu} = X\mathbf{w}$.

Zero step: $\boldsymbol{\mu}_0 = \mathbf{0}$, the residual vector $\varepsilon_0 = \mathbf{y} - \boldsymbol{\mu}_0$.

First step: $corr(\mathbf{y}, \mathbf{x}_1) > corr(\mathbf{y}, \mathbf{x}_2)$, then $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + w_1\mathbf{x}_1$,
where w_1 provides $\mathbf{y} - \boldsymbol{\mu}_1$ to lie on the bisecting line between $\mathbf{x}_1, \mathbf{x}_2$.

Second step: the parameter w_2 satisfies

$$\boldsymbol{\mu}_2 = \boldsymbol{\mu}_1 + w_2\mathbf{u}_2 = \mathbf{y}$$

for $m = 2$, where $\mathbf{u}_2$ is the unit vector.

2. Delete a feature: Belsley method

Singular value decomposition of the correlation matrix $X^T X$: $\quad X^T X = U \Lambda V^T,$

where Λ is the diagonal matrix with singular values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Condition indices : $\quad \eta_j = \frac{\lambda_{max}}{\lambda_j}$.

The variance of w_i : $\sigma^{-2}\mathcal{V}(w_i) = (q_{i1} + q_{i2} + \cdots + q_{in}) \sum_{j=1}^{n} \frac{v_{ij}^2}{\lambda_{ij}^2}$,
q_{ij} is the contribution of corresponding summand.

$$\hat{j} = \arg\max \eta_j$$

Condition indices	$\mathcal{V}(w_1)$	$\mathcal{V}(w_2)$	$\dots$	$\mathcal{V}(w_n)$
$\eta_{\hat{j}}$	$q_{1\hat{j}}$	$q_{2\hat{j}}$	$\dots$	$q_{n\hat{j}}$

Correlated features correspond large $q_{k\hat{j}}$.

Experiment: historical data of European options

- $\mathbf{K} = \{K_1, K_2, \dots, K_9\} = \{1400, 1425, ..., 1575, 1600\}$ is the set of strike prices,
- $t = \{t_1, t_2, \dots, t_{36}\}$ is the set of time ticks (Maturity),
- $C_{K,t}$ is the historical option price,
- P_t is the historical security price.

The sample set for regression analysis

$$\{(\mathbf{x}_i, y_i)\}_{i=1}^m = \{(\langle K_i, t_i \rangle, \sigma_i)\}_{i=1}^m.$$

Set of primitives $G = \{\frac{1}{x}, \sqrt{x}, \ln(x), \tanh(x)\}$.

Index mapping

Maturity	$C_{K_1,t}$	$C_{K_2,t}$	...	$C_{K_8,t}$	$C_{K_9,t}$	Price
-129	129.70	109.00	...	17.60	9.10	1495.42
-128	129.70	109.10	...	18.00	10.10	1494.25
...	...	...	...	...	...	...
-1	90	64.3	...	0.7	0.25	1473.99

Implied volatility

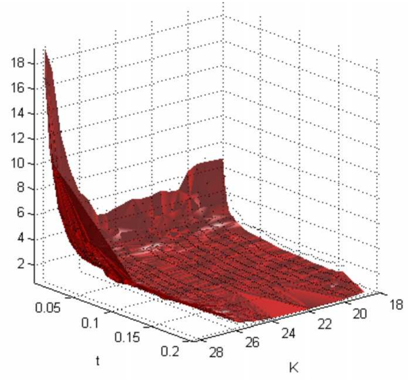
$$\sigma_i = arg \min_{\sigma \in [0,1.5]} \left(C_{K_i,t_i} - C(\sigma, P_{t_i}, B, K_i, t_i) \right),$$

where

$$(K_k.t_\tau) = (K_i, t_i) \in \mathbf{K} \times t,$$

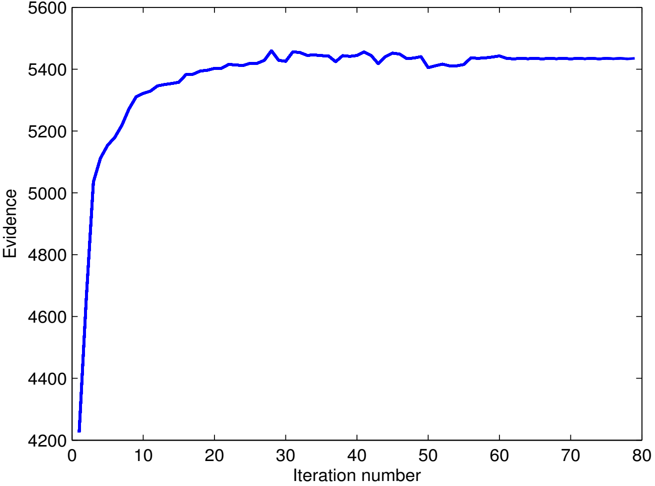
$$i = \tau + k(|T| - 1).$$

Example of the resulting model

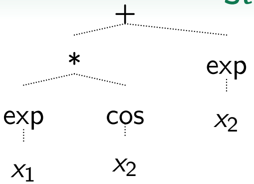


$$w_1 + w_2 t^{\frac{1}{2}} \ln K + w_3 \ln K \ln t + w_4 K^{\frac{1}{2}} \ln^2 K + w_5 K^{-\frac{1}{2}} t^{-2} +$$
$$+ w_6 \ln^2 K \tanh K + w_7 K^{-2} \ln K + w_8 t \ln K + w_9 t^{\frac{1}{2}} \ln K \ln t +$$
$$+ w_{10} \ln K \ln^2(t) + w_{11} \ln K \tanh^2 t + w_{12} K^{-3} + w_{13} K^{-1} t^{-1} \tanh K.$$

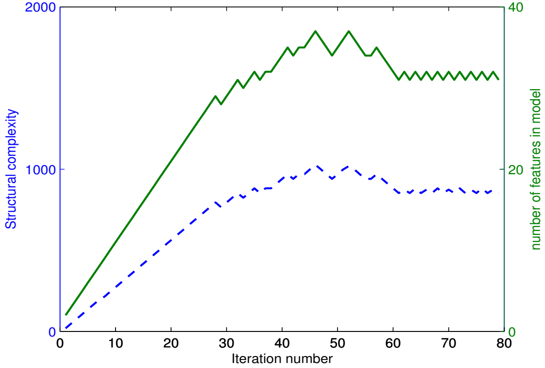
Evidence convergence



Structural complexity



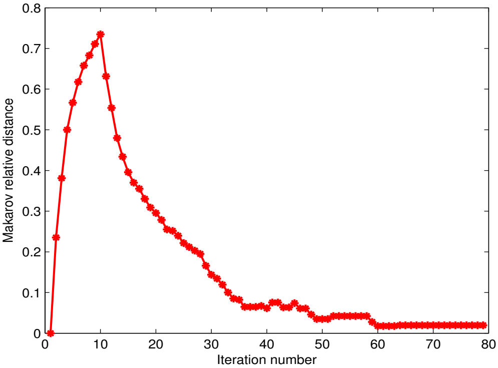
The model is represented as a tree.
The structural complexity is a sum of the number of nodes in all subtrees of given tree.



Distance between models

Distance measure is determined by number of nodes p_{12} in maximum common subgraph of two trees T^1 and T^2.
Let p_1 and p_2 — numbers of nodes in trees trees T^1 and T^2.
Distance measure:

$$r_{12} = (p_1 + p_2 - 2p_{12})/(p_1 + p_2).$$



Results of comparison

Algorithm	CV	AIC	BIC	C_p	$\lg \kappa$	k
Genetics	0,107	-1152	-1072	337	13	26
GMDH	0,194	-1076	-1045	745	6	10
Stepwise	0,154	-1092	-1055	644	7	12
Ridge	0,146	-819	-330	832	33	160
Lasso	0,147	-1089	-1034	611	5	18
Stagewise	0,096	-1157	-1077	324	9	26
FOS	0,135	-1105	-1044	527	7	20
LARS	0,095	-1102	-1017	492	7	28
Proposed	0,123	-1118	-1054	469	2	21

$$AIC = m \ln(S/m)) + 2k, \quad BIC = m \ln(S/m)) + k \ln m,$$

$$C_p = S/\sigma^2 - 2k + m.$$

Conclusion

- The feature generation method was proposed.
- The proposed algorithm construct the model in the maximum evidence neighborhood.
- This algorithm allows to get more stable models in comparison with well-known algorithms.

The proposed algorithm is similar to stepwise regression.
But instead of common criterion (Mallow's C_p) the algorithm uses the Coherent Bayesian Inference. This allows to obtain the model plausible given the data.