Indices construction using linear and ordinal expert estimations

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There are lots of ways to construct indices. However, when algorithms are chosen and some results obtained, the following question arises:

- **How to show adequacy of the calculated indices?**

To answer the question analysts invite experts. The experts express their opinion and then the second question arises:

- **How to show that expert estimations are valid?**

What is an index?

- There is a set of objects, i.e. power plants:
  - Beckjord
  - East Bend
  - Miami Fort
  - Zimmer
- Index a measure of an object's quality,
  a scalar corresponded to an object.
- Expert estimation of an object's quality
  could be an index, too.

How to construct an index?

- Assign a comparison criterion
- Gather a set of comparable objects
- Gather features of the objects
- Make a data table: objects/features, i.e.

#	Plant Name	Plant Type	Total Net Generation	CO_2 emission	NO_X emission	SO_X emission	Population density
			10^6 KWHours	Short tons per month	Short tons per month	Short tons per month	Qty per sq.mile
1	**Beckjord**	Coal	458505	191	16	45	23
3	**East Bend**	Coal	356124	147	16	43	34
4	**Miami Fort**	Coal	484590	204	6	23	45
5	**Zimmer**	Coal	818435	329	5	64	34
		Optimal value	max	min	min	min	min

The criterion could be: **Ecological footprint of a plant**

Notations

$A=\{a_{ij}\}$ – $(n\times m)$ real matrix, **data set**,

$\mathbf{q}=[q_1,\ldots,q_m]^{\mathrm{T}}$ – vector of **objects' indices**,

$\mathbf{w}=[w_1,..,w_n]^{\mathrm{T}}$ – vector of **features' importance weights**,

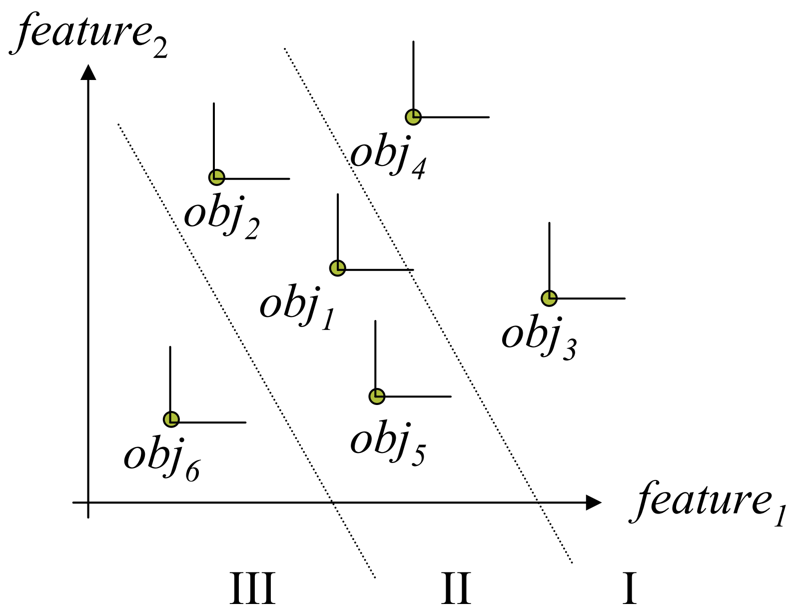
$\mathbf{q}_0$, $\mathbf{w}_0$ – expert estimations of indices and weights.

Usually, data prepared so that

1. the minimum of each feature equals 0, while the maximum equals 1;
2. the bigger value of each implies better quality of the index.

The first method, Pareto slicing

An easiest method to obtain indices in ordinal scales is to find non-dominated objects at each slicing level.

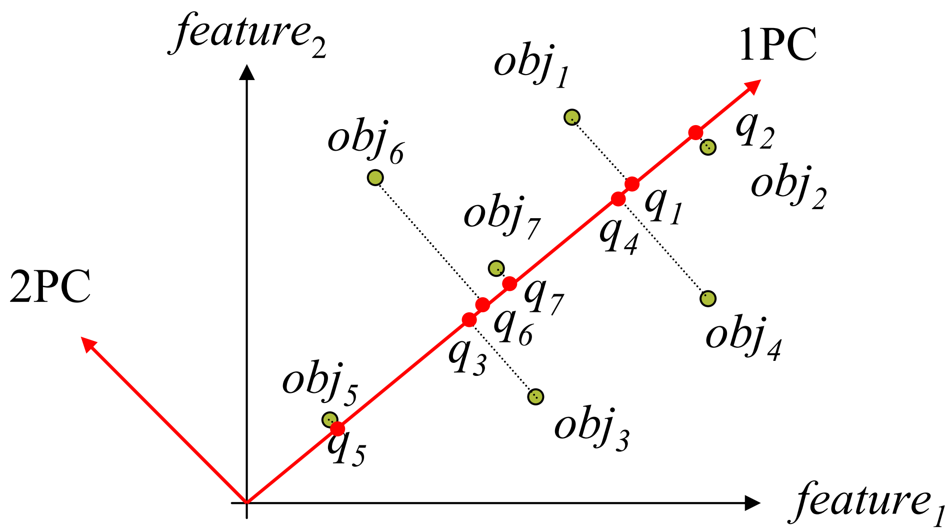


Unsupervised way,

Principal Components Analysis

Z=AW, where W—rotation matrix of the principal components.

$\mathbf{q}_{PCA}=A\mathbf{w}_{1PC}$, where $\mathbf{w}_{1PC}$ is the 1st column vector of W.



PCA gives minimal mean square error between objects and their projections.

Supervised way-1, the Weighted sum

$$\mathbf{q}_1 = A\,\mathbf{w}_0.$$

Supervised way-2,

the Expert-Statistical Technique

$\mathbf{w}_1 = \arg\min \|\mathbf{q}_0 - A\,\mathbf{w}_1\|$,

or

$\mathbf{w}_1 = A^+\,\mathbf{q}_0$.

$A^+ = VW^{-1}U^T$, while $A = UWV^T$ – singular values decomposition.

The problem of concordance

- **We have**

The data table A

Expert estimations $\mathbf{q}_0, \mathbf{w}_0$.

Calculated weights and indices $\mathbf{q}_1, \mathbf{w}_1$.

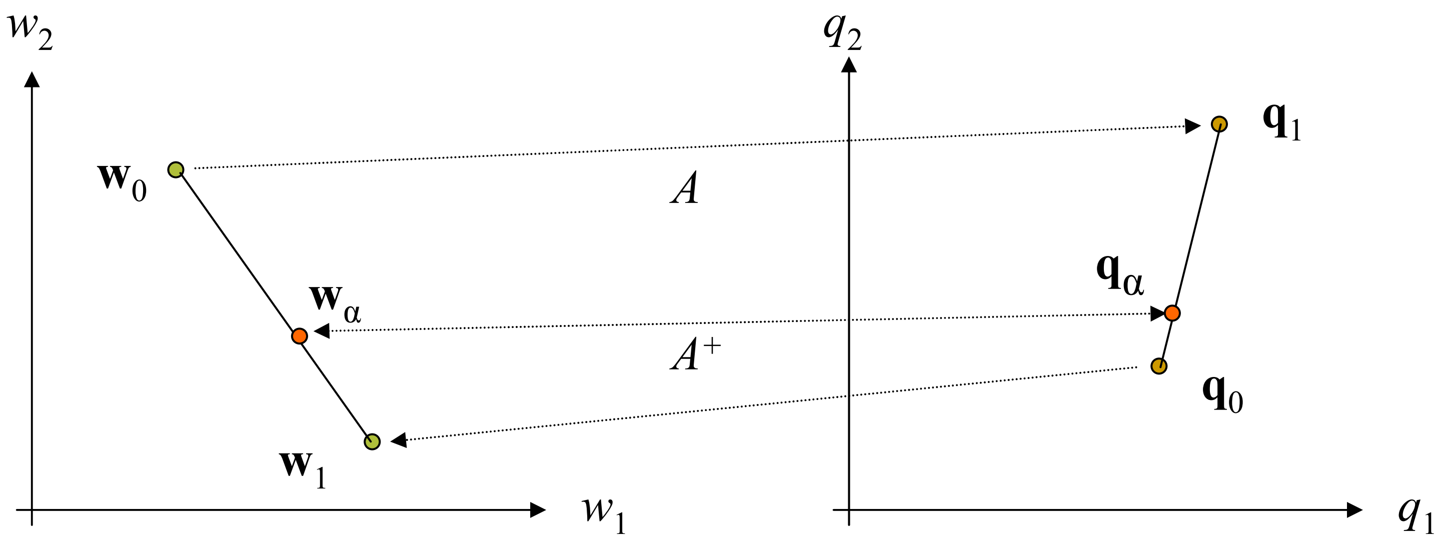
- **Dilemma**

Calculated indices are not the same as the expert estimations for the indices;

as well, calculated weights are not the same as the expert estimations of the weights:

in general, neither $\mathbf{q}_0 \neq A\mathbf{w}_0$, nor $\mathbf{w}_0 \neq A^+\mathbf{q}_0$.

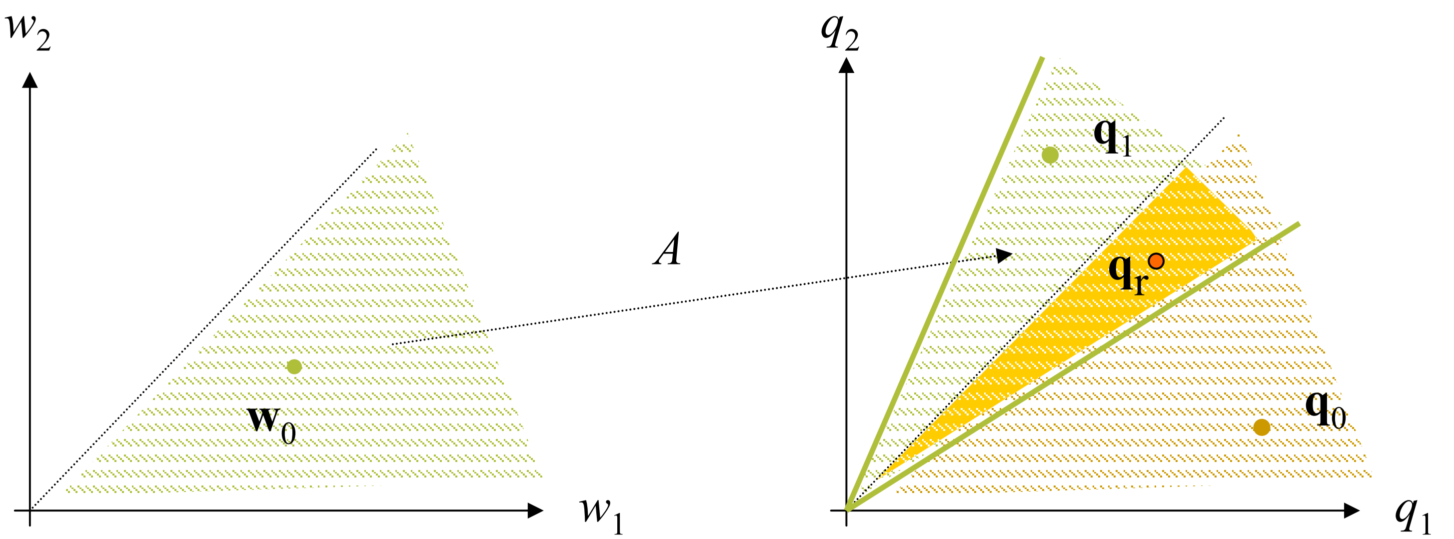
Linear concordance procedure



$$\mathbf{w}_\alpha = \alpha A^{+}\mathbf{q}_0 + (\mathbf{1}-\alpha)\mathbf{w}_0, \qquad \mathbf{q}_\alpha = (1-\alpha)A\mathbf{w}_0 + \alpha\mathbf{q}_0.$$

Parameter α is in [0,1].
α = 0, then we trust expert estimations of the weights
α = 1, then we trust expert estimations of the indices

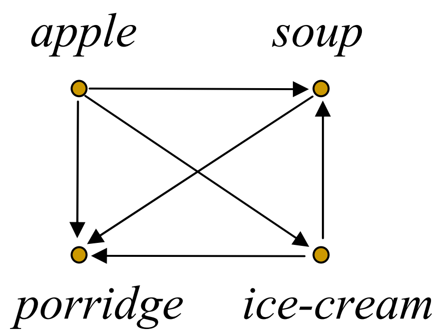
Ordinal concordance procedure

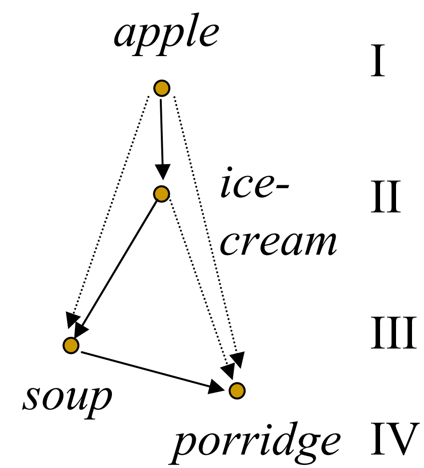


$$\mathbf{w}_0 = [w_1 > w_2 > \ldots > w_n]^{\mathrm{T}}, \quad \mathbf{q}_0 = [q_1 > q_2 > \ldots > q_m]^{\mathrm{T}}.$$

Pair-wise comparison

	a	*s*	*p*	*i-c*
apple	●	+	+	+
soup		●	+	–
porridge			●	–
ice-cream				●





If an object in a row is better than the other one in a column then put “+”, otherwise “-”.

Make a graph, *row* + *column* means *row* ●→● *column*.

Find the top and remove extra nodes.

Results of the concordance are

- adequate indices.
- reasoned expert estimations.
- We know why expert valued an object
- and what contribution a feature makes to an index.

And we have precise weights

to make indices in the future.

Thanks!

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