Specification of linear and rank-scaled expert estimations using measured data

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What is the index?

- There is a set of objects, i.e. power plants:
  - Beckjord
  - East Bend
  - Miami Fort
  - Zimmer
- The index is a measure of an object's quality.
  It is a scalar, corresponded to an object.
- Expert estimation of an object's quality
  could be an index, too.

Examples

Index name	**Objects**	**Features**	**Model**
TOEFL	Students	Tests	Sum of scores
Eurovision	Singers	Televotes, Jury votes	Linear (weighted sum)
S&P500, NASDAQ	Time ticks	Shares (prices, volumes)	Non-linear
Bank ratings	Banks	Requirements	By an expert commission
Kyoto-index	**Power plants**	**Greenhouse gases**	**Linear**

There are lots of ways to construct indices. However, when algorithms are chosen and some results obtained, the following question arises:

- **How to show adequacy of the calculated indices?**

To answer the question analysts invite experts. The experts express their opinion and then the second question arises:

- **How to show that expert estimations are valid?**

How to construct an index?

- Assign a comparison criterion.
- Gather a set of comparable objects.
- Gather features of the objects.
- Make a data table: objects/features, i.e.

#	Plant Name	Plant Type	Total Net Generation	CO_2 emission	NOx emission	SOx emission	Population density
			10^6 KWHours	Short tons per month	Short tons per month	Short tons per month	Qty per sq.mile
1	**Beckjord**	Coal	458505	191	16	45	23
3	**East Bend**	Coal	356124	147	16	43	34
4	**Miami Fort**	Coal	484590	204	6	23	45
5	**Dark Creek**	Coal	818435	329	5	64	34
	Optimal value		**max**	**min**	**min**	**min**	**min**

The criterion could be: **Ecological footprint of a plant**

Notations

$A=\{a_{ij}\}$ – $(n \times m)$ real matrix, **data set**,

$\mathbf{q}=[q_1, \ldots, q_m]^T$ – vector of **object indices**,

$\mathbf{w}=[w_1, \ldots, w_n]^T$ – vector of

feature importance weights,

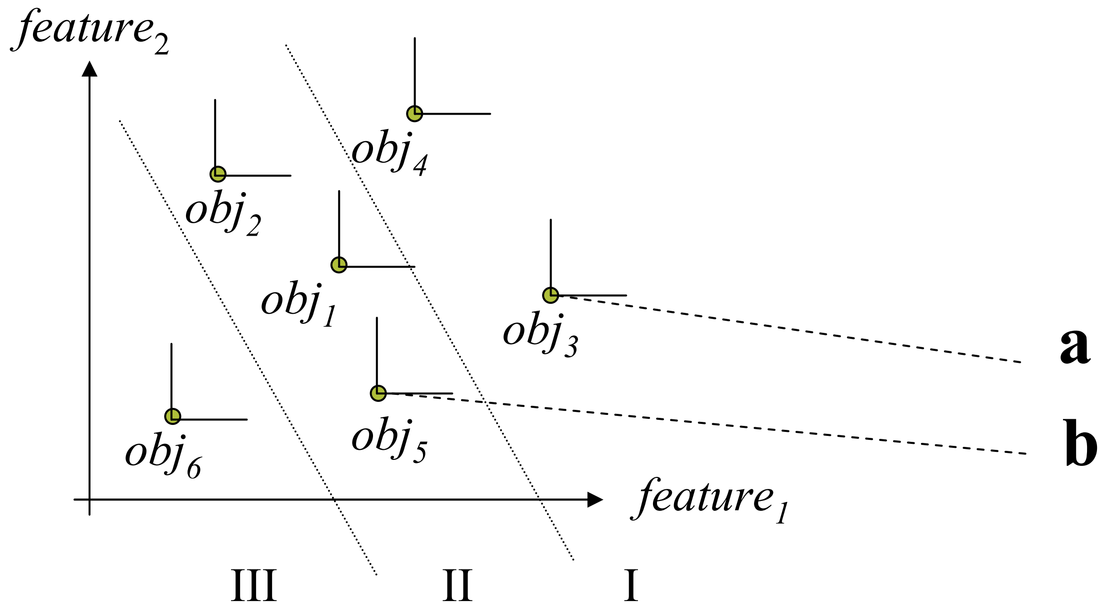
$\mathbf{q}_0, \mathbf{w}_0$ – **expert estimations of indices and weights**.

Usually, data prepared so that

1. the minimum of each feature equals 0, while the maximum equals 1;
2. the bigger value of each implies better quality of the index.

The first method, Pareto slicing

An easiest method to obtain indices in ordinal scales is to find non-dominated objects at each slicing level.



The object **a** is non-dominated if there is no **b**$_i$ such that $b_{ij} \geq a_i$ for all features j.

Supervised way-1,

the Weighted sum

$$\mathbf{q}_1 = A\,\mathbf{w}_0.$$

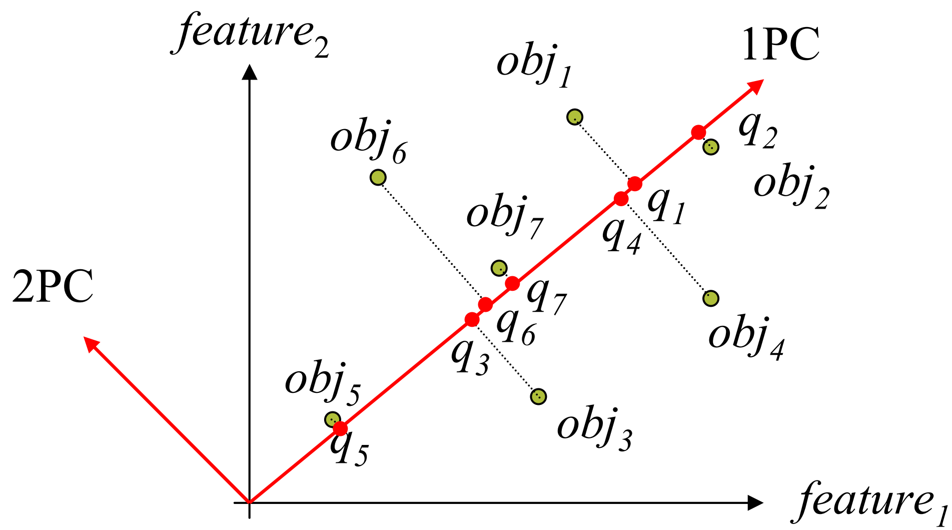
	w_1	$\dots$	w_n
q_1	a_{11}	$\dots$	a_{1n}
$\dots$	$\dots$	$\dots$	$\dots$
q_m	a_{m1}	$\dots$	a_{mn}

Unsupervised way,

Principal Components Analysis

$Q=AW$, where W—rotation matrix of the principal components.

$\mathbf{q}_{PCA}=A\mathbf{w}_{1PC}$, where $\mathbf{w}_{1PC}$ is the 1st column vector of W.



PCA gives minimal mean square error between objects and their projections.

Unsupervised way,

useful tool for PCA

$$A = ULW^T$$

$$A^T A = WLU^T ULW^T$$

$$A^T AW = WL^2$$

Supervised way-2,

the Expert-Statistical Technique

$$\mathbf{w}_1 = \arg\min \|\mathbf{q}_0 - A\,\mathbf{w}\|^2,$$

least squares, $\mathbf{w}_1 = (A^T A)^{-1} A^T \mathbf{q}_0.$

The problem of specification

- **We have**

the data table A,

expert estimations $\mathbf{q}_0$, $\mathbf{w}_0$,

calculated weights and indices $\mathbf{q}_1$, $\mathbf{w}_1$.

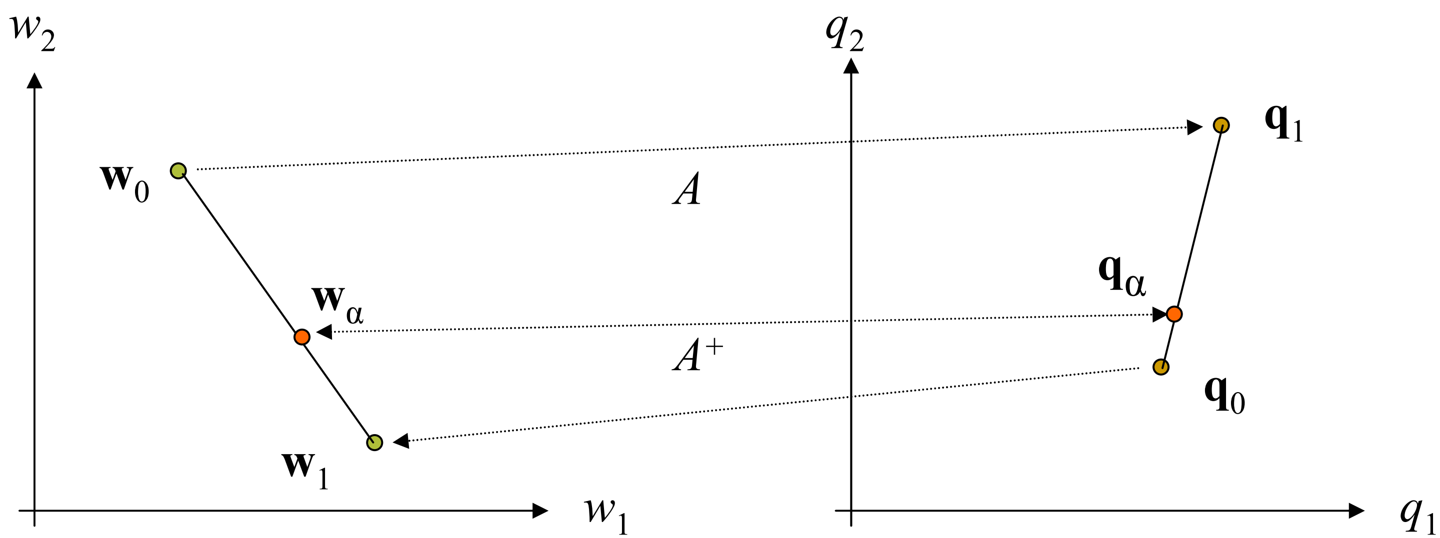
- **Contradiction**

Calculated indices are not the same as the expert estimations for the indices;

as well, calculated weights are not the same as the expert estimations of the weights:

in general, neither $\mathbf{q}_0 \neq A\mathbf{w}_0$, nor $\mathbf{w}_0 \neq A^{+}\mathbf{q}_0$.

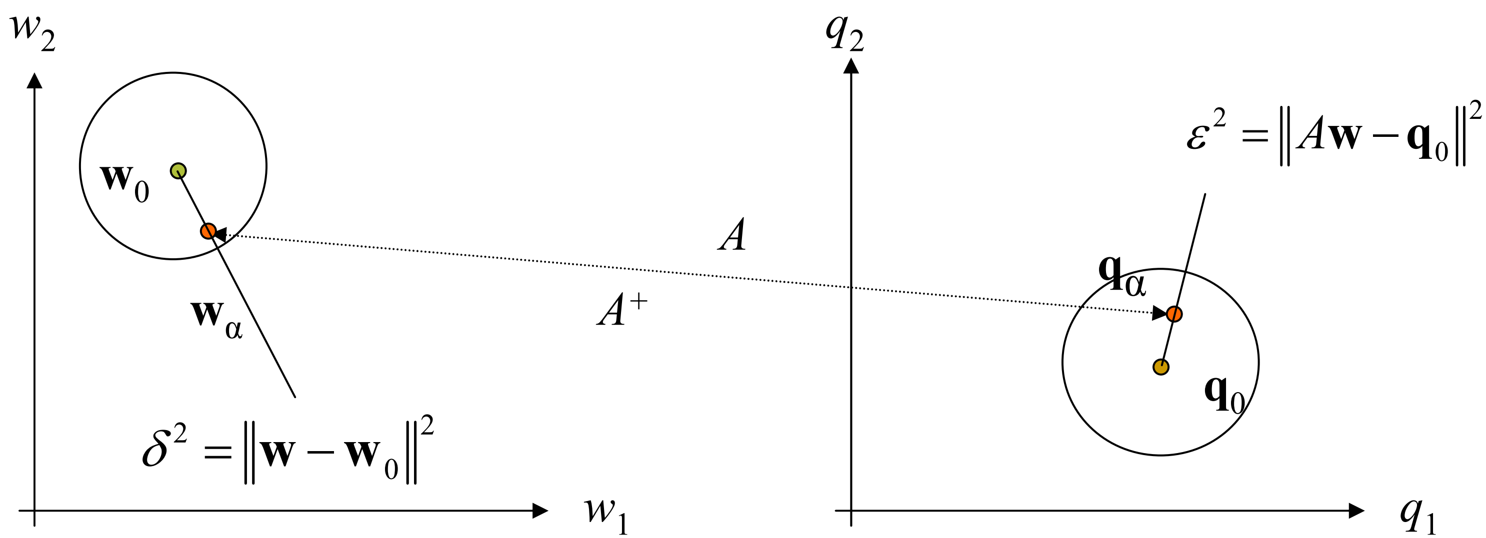
Linear specification



$$\mathbf{w}_\alpha = \alpha A^+ \mathbf{q}_0 + (\mathbf{1}\text{-}\alpha)\mathbf{w}_0, \qquad \mathbf{q}_\alpha = (1\text{-}\alpha) A \mathbf{w}_0 + \alpha \mathbf{q}_0.$$

Parameter α is in [0,1].
α = 0, we trust expert estimations of the weights,
α = 1, we trust expert estimations of the indices.

Quadratic specification

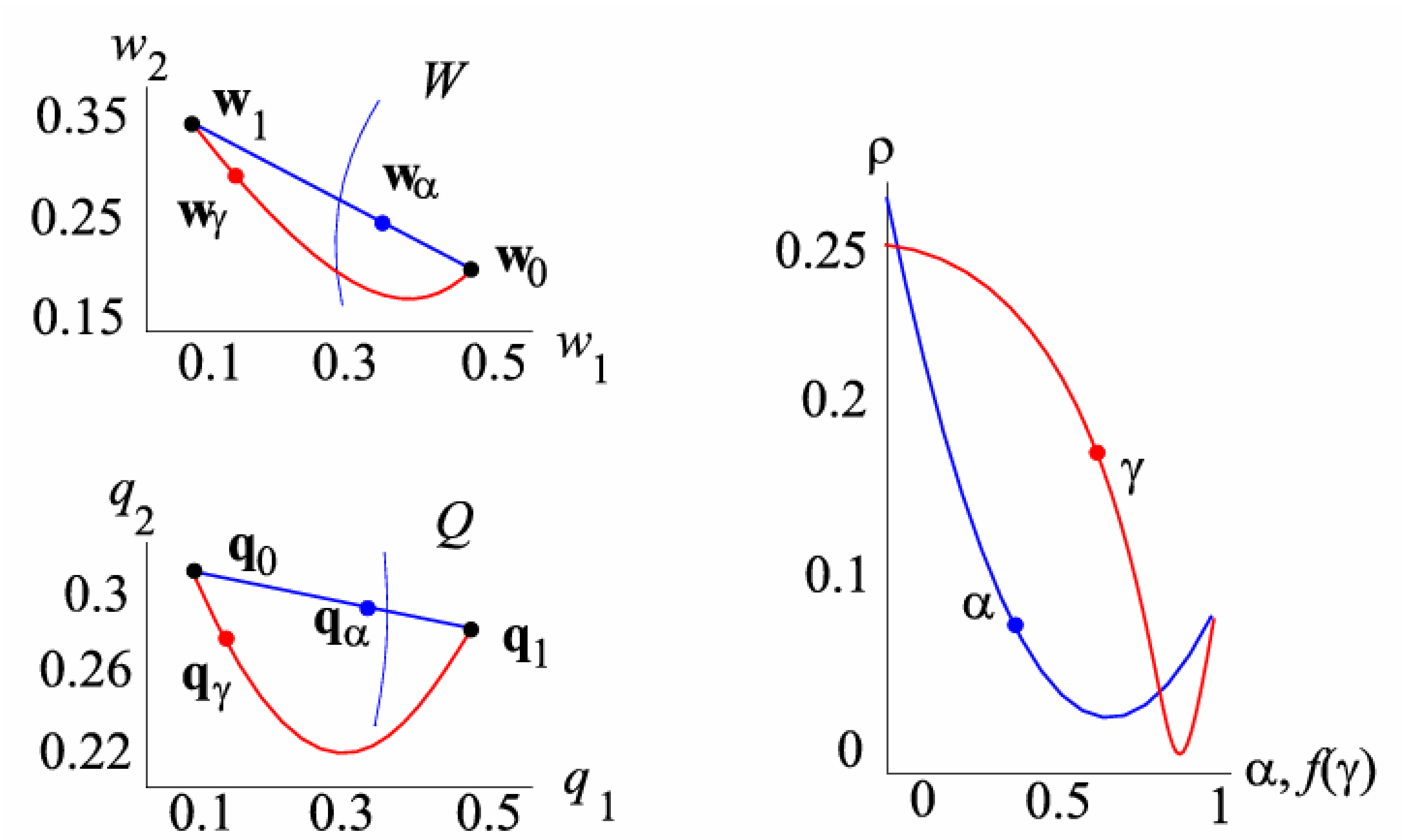


$$\mathbf{w}_\gamma = \arg\min_{\mathbf{w}\in W}(\varepsilon^2 - \gamma^2\delta^2), \quad \mathbf{w}_\gamma = (A^T A + \gamma^2 I)^{-1}(A^T\mathbf{q}_0 + \gamma^2\mathbf{w}_0).$$

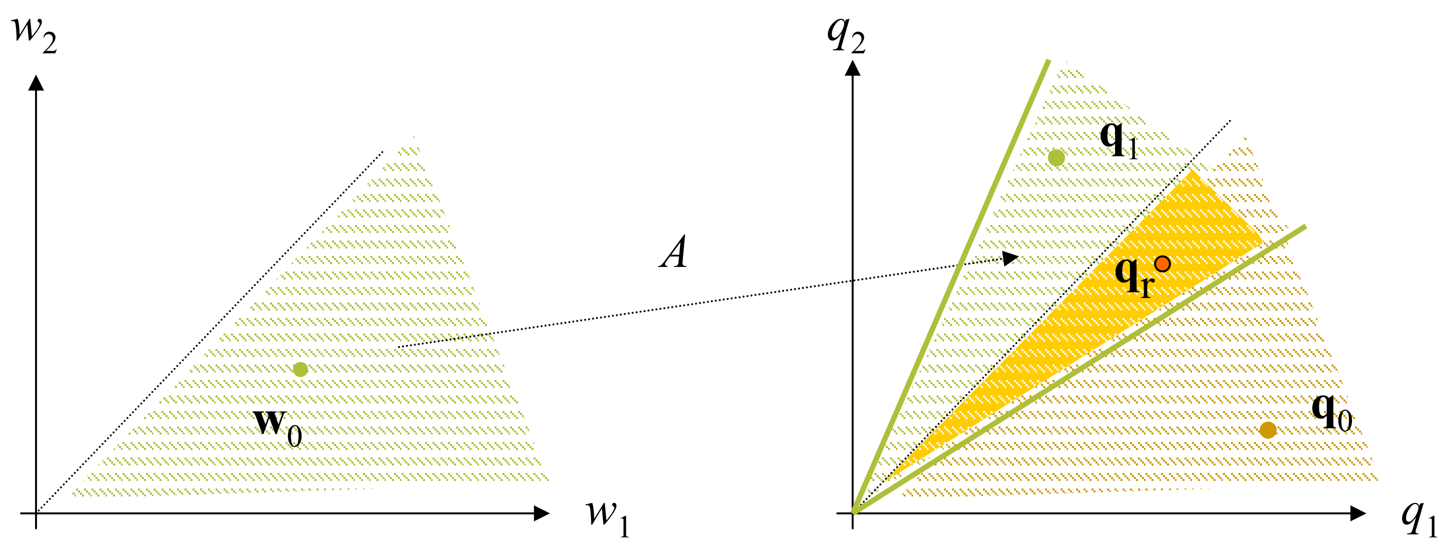
If parameter γ^2 is 0, then we trust expert estimations of the indices.

Comparison of the methods,

what is the difference?



Ordinal specification



$$\mathbf{w}_0 = [w_1 \geq w_2 \geq \ldots \geq w_n \geq 0]^T, \mathbf{q}_0 = [q_1 \geq q_2 \geq \ldots \geq q_m \geq 0]^T.$$

Rank-scaled expert estimations

$$\mathbf{w}_0 = [w_1 \geq w_2 \geq \ldots \geq w_n \geq 0]^T, \mathbf{q}_0 = [q_1 \geq q_2 \geq \ldots \geq q_m \geq 0]^T.$$

$$Q_0 = \{\mathbf{q}_0 \mid J_m \mathbf{q}_0 \geq \mathbf{0}\},$$
$$W_0 = \{\mathbf{w}_0 \mid J_n \mathbf{w}_0 \geq \mathbf{0}\}.$$

$$J = \begin{pmatrix} 1 & -1 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

The cones intersection exists

$$\mathbf{q}_1 \in AW_0 \cap Q_0,$$

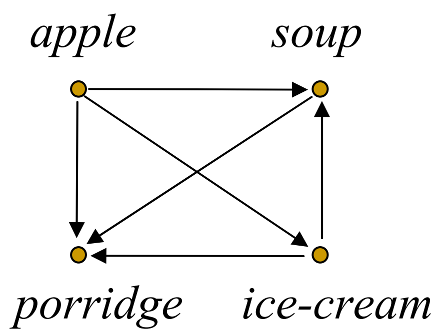
or not, then specify

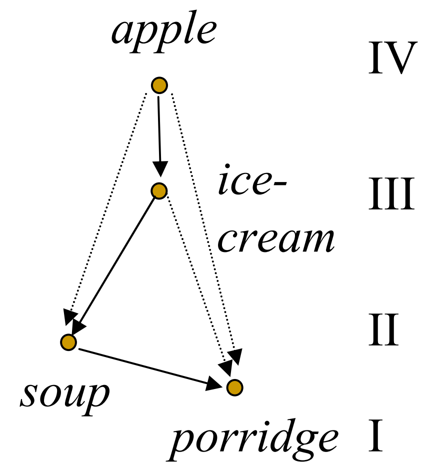
$$\mathbf{q}_\alpha = (1-\alpha)A\mathbf{w}' + \alpha\mathbf{q}', \quad \text{where}$$

$$\mathbf{w}', \mathbf{q}' = \arg \min_{\substack{\mathbf{w}\in W_0, \|\mathbf{w}\|^2=1 \\ \mathbf{q}\in Q_0, \|\mathbf{q}\|^2=1}} \|A\mathbf{w} - \mathbf{q}\|^2 .$$

Pair-wise comparison

	a	*s*	*p*	*i-c*
apple	•	+	+	+
soup		•	+	–
porridge			•	–
ice-cream				•





If an object in a row is better than the other one in a column then put "+", otherwise "-".

Make a graph, *row* + *column* means *row* •→• *column.*

Find the top and remove extra nodes.

The results of the specification are

- adequate indices,
- reasoned expert estimations.

We know why our expert valued each object
and what contribution each feature makes to the index.

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List of the constructed indices

1. Integral indicators of the quality of life in the Russian regions
2. Human development index in Russia
3. Kyoto-index: power plant ecological footprints in the USA, Ohio
4. Protected area management effectiveness in Russia
5. Index of rare and Red List species in Russia
6. Econometrical index of the Russian economy state
7. The high school science effectiveness for the Ministry of Education
8. Croatian power plant ecological footprints